PQIC Mathematical Notes, Analytical Series, Number 79:
Univariate Continuous F Distribution Random Variate Generation Using Distribution Ratio Methodology

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Abstract

The purpose of this memorandum is to document the analytical methods by which random variates from a univariate continuous F probability distribution may be generated regardless of the relative sizes of its parameters, and to document alternative calculation methods for estimating F distribution values when one or the other (or both) positive parameters increase without bound.

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1. Introduction

The univariate continuous F probability distribution is used within PQICSTAT™ as the discriminator of choice for reducing designed experiment linear models to the fundamental core of significant factors based on a given dataset. The iterative steps of re-allocating the sum of squares to reflect the annexation of non-significant factors to the error term requires repeated calculation of an F distributed statistic with increasing second degree of freedom. To control the extent to which factors are accepted or rejected in this process, it is important to understand the power of these decisions, which must be evaluated through simulated results, i.e., through the generation of large numbers of random variates, to study the behavior of the F distributions under a variety of alternative hypotheses. In this respect, random variates from the F distribution must not only be available for simulated results in this and other contexts, but also asymptotic estimation is needed when the degrees of freedom become very large in extensive datasets.

The purpose of this memorandum is to document the analytical methods by which random variates from a univariate continuous F probability distribution may be generated regardless of the relative sizes of its parameters, and to document alternative calculation methods for estimating the F distribution values when one or the other (or both) positive parameters increase without bound.

Definition 1 *A univariate continuous random variable X is said to have an* F Distribution With $r > 0$ and $s > 0$ Degrees Of Freedom, *symbolized as $F(r,s)$, if r and s are integers, and its density function $f_X(x;r,s)$ is given by*

$$f_X(x;r,s) = \frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)} \left(\frac{r}{s}\right)^{\frac{r}{2}} x^{\frac{r}{2}-1}\left(1+\frac{r}{s}x\right)^{-\frac{r+s}{2}},\ x \geq 0$$

Claim 2 *The function $f_X(x;r,s)$ is a density function of a probability distribution.*

Proof. For r even, we have

$$\begin{aligned}
\int_0^\infty f_X(x;r\text{ even},s)\,dx &= \frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)} \left(\frac{r}{s}\right)^{\frac{r}{2}} \int_0^\infty x^{\frac{r}{2}-1}\left(1+\frac{r}{s}x\right)^{-\frac{r+s}{2}} dx \\
&= \frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)} \left(\frac{r}{s}\right)^{\frac{r}{2}} \left(\begin{array}{c} \frac{s}{r}x^{\frac{r}{2}-1}\frac{\left(1+\frac{r}{s}x\right)^{1-\frac{r+s}{2}}}{1-\frac{r+s}{2}}\Bigg|_{x=0}^{x\to\infty} \\ -\left(\frac{s}{r}\right)\frac{\left(\frac{r}{2}-1\right)}{1-\frac{r+s}{2}}\int_0^\infty x^{\frac{r}{2}-2}\left(1+\frac{r}{s}x\right)^{1-\frac{r+s}{2}} dx \end{array}\right), \\
&\left(\begin{array}{c} u = x^{\frac{r}{2}-1},\ dv = \left(1+\frac{r}{s}x\right)^{-\frac{r+s}{2}} dx \\ du = \left(\frac{r}{2}-1\right)x^{\frac{r}{2}-2}\,dx,\ v = \left(\frac{s}{r}\right)\frac{\left(1+\frac{r}{s}x\right)^{1-\frac{r+s}{2}}}{1-\frac{r+s}{2}} \end{array}\right) \\
&= \frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)} \left(\frac{r}{s}\right)^{\frac{r}{2}} \left(\begin{array}{c} \left(2\frac{s}{r(2-(r+s))}\lim_{x\to\infty}\frac{\left(\frac{x}{1+\frac{r}{s}x}\right)^{\frac{r}{2}-1}}{\left(1+\frac{r}{s}x\right)^{\frac{s}{2}}} = 0 \text{ since } s>0\right) \\ +\left(\frac{s}{r}\right)\frac{\frac{r}{2}-1}{\frac{r+s}{2}-1}\int_0^\infty x^{\frac{r}{2}-2}\left(1+\frac{r}{s}x\right)^{1-\frac{r+s}{2}} dx \end{array}\right) \\
&= \frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)} \left(\frac{r}{s}\right)^{\frac{r}{2}-1} \frac{\frac{r}{2}-1}{\frac{r+s}{2}-1}\int_0^\infty x^{\frac{r}{2}-2}\left(1+\frac{r}{s}x\right)^{1-\frac{r+s}{2}} dx \\
&= \cdots \\
&= \frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)} \left(\frac{r}{s}\right)^{\frac{r}{2}-\left(\frac{r}{2}-1\right)} \left(\begin{array}{c} \left(\frac{\frac{r}{2}-1}{\frac{r+s}{2}-1}\right)\cdots \\ \times\left(\frac{\frac{r}{2}-\left(\frac{r}{2}-1\right)}{\frac{r+s}{2}-\left(\frac{r}{2}-1\right)}\right) \end{array}\right) \int_0^\infty x^{\frac{r}{2}-\frac{r}{2}}\left(1+\frac{r}{s}x\right)^{\left(\frac{r}{2}-1\right)-\frac{r+s}{2}} dx \\
&= \frac{s}{2}\left(\frac{\left(\frac{r}{2}-1\right)!}{\Gamma\left(\frac{r}{2}\right)}\right)\left(\frac{\Gamma\left(\frac{r+s}{2}\right)}{\frac{s}{2}\Gamma\left(\frac{s}{2}\right)\prod_{k=1}^{\frac{r}{2}-1}\left(\frac{r+s}{2}-k\right)}\right)\int_0^\infty \frac{r}{s}\left(1+\frac{r}{s}x\right)^{-\frac{s}{2}-1} dx \\
&= -\frac{\frac{s}{2}}{\frac{s}{2}}\left(1+\frac{r}{s}x\right)^{-\frac{s}{2}}\Bigg|_{x=0}^{x\to\infty}
\end{aligned}$$

$$= 1, \text{ since } s > 0$$

When r is odd, then $r-1$ is even, so we have[1]

$$\begin{aligned}
\int_0^\infty f_X(x; r \text{ odd}, s)\, dx &= \frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)} \left(\frac{r}{s}\right)^{\frac{r}{2}} \int_0^\infty x^{\frac{r}{2}-1} \left(1 + \frac{r}{s}x\right)^{-\frac{r+s}{2}}\, dx \\
&= \text{ same steps as for } r \text{ even} \\
&= \left(\begin{array}{c} \frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)} \left(\frac{r}{s}\right)^{\frac{r}{2}-\left(\frac{r-1}{2}-1\right)} \left(\frac{\frac{r}{2}-1}{\frac{r+s}{2}-1}\right) \cdots \\ \times \left(\frac{\frac{r}{2}-\left(\frac{r-1}{2}-1\right)}{\frac{r+s}{2}-\left(\frac{r-1}{2}-1\right)}\right) \int_0^\infty x^{\frac{r}{2}-1-\left(\frac{r-1}{2}-1\right)} \left(1+\frac{r}{s}x\right)^{\left(\frac{r-1}{2}-1\right)-\frac{r+s}{2}}\, dx \end{array} \right) \\
&= \left(\begin{array}{c} \frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)} \left(\frac{r}{s}\right)^{\frac{3}{2}} \left(\frac{\frac{r}{2}-1}{\frac{r+s}{2}-1}\right) \cdots \\ \times \left(\frac{\frac{3}{2}}{\frac{r+s}{2}-\left(\frac{r-1}{2}-1\right)}\right) \int_0^\infty x^{\frac{1}{2}} \left(1+\frac{r}{s}x\right)^{-\frac{s+3}{2}}\, dx \end{array} \right) \\
&= \left(\begin{array}{c} \frac{2}{\sqrt{\pi}} \left(\frac{\left(\Gamma\left(\frac{r+s}{2}\right)=\Gamma\left(\frac{s}{2}+\frac{3}{2}\right) \prod_{j=1}^{\frac{r-1}{2}-1} \left(\frac{r+s}{2}-j\right)\right)}{\Gamma\left(\frac{s}{2}\right) \prod_{k=1}^{\frac{r-1}{2}-1} \left(\frac{r+s}{2}-k\right)} \right) \left(\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \prod_{k=1}^{\frac{r-1}{2}-1} \left(\frac{r}{2}-k\right)}{\Gamma\left(\frac{r}{2}\right)} \right) \\ \times \int_0^\infty \frac{r}{s} \left(\frac{r}{s}x\right)^{\frac{1}{2}} \left(1+\frac{r}{s}x\right)^{-\frac{s+3}{2}}\, dx \end{array} \right) \\
&= \frac{4}{\sqrt{\pi}} \frac{\Gamma\left(\frac{s+3}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \int_0^{\frac{\pi}{2}} \tan^2 u \sec^2 u \left(1+\tan^2 u\right)^{-\frac{s+3}{2}}\, du, \\
&\left(\begin{array}{c} \frac{r}{s}x = \tan^2 u, \\ dx = 2\frac{s}{r} \tan u \sec^2 u\, du \end{array} \right) \qquad (1) \\
&= \frac{4}{\sqrt{\pi}} \frac{\Gamma\left(\frac{s+3}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \int_0^{\frac{\pi}{2}} \left(\sec^2 u - 1\right) \sec^{-s-1} u\, du \\
&= \frac{4}{\sqrt{\pi}} \frac{\Gamma\left(\frac{s+3}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \left(\int_0^{\frac{\pi}{2}} \cos^{s-1} u\, du - \int_0^{\frac{\pi}{2}} \cos^{s+1} u\, du \right) \\
&= \frac{4}{\sqrt{\pi}} \frac{\Gamma\left(\frac{s+3}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \left(\int_0^{\frac{\pi}{2}} \cos^{s-1} u\, du - \int_0^{\frac{\pi}{2}} \cos^{(s+2)-1} u\, du \right) \\
&= \frac{4}{\sqrt{\pi}} \frac{\Gamma\left(\frac{s+3}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \left(\frac{1}{2}\sqrt{\pi} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} - \frac{1}{2}\sqrt{\pi} \frac{\Gamma\left(\frac{s+2}{2}\right)}{\Gamma\left(\frac{s+3}{2}\right)} \right) \qquad (2) \\
&= 2 \left(\frac{\Gamma\left(\frac{s+3}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} - \frac{\Gamma\left(\frac{s+2}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \right) \\
&= 2 \left(\frac{s+1}{2} - \frac{s}{2} \right) \\
&= 1
\end{aligned}$$

■

[1] The step labeled (2) in this proof was proven in Claim 2 in *PQIC Mathematical Notes, Analytical Series, Number 78: Univariate Continuous Student-T Distribution Random Variate Generation Using Mixtures Methodology*}, December 2020, where, for integer n, we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-1} \psi\, d\psi = 2 \int_0^{\frac{\pi}{2}} \cos^{n-1} \psi\, d\psi$$

since the cosine is an even function.

2. Main Theorem

Theorem 3 *If* $X_i \sim \Gamma\left(\frac{\alpha_i}{2}, \frac{\beta_i}{2}\right)$, $i = 1, 2$, $\alpha_i, \beta_i > 0$, *are independent, then* $\frac{\alpha_2 \beta_1 X_1}{\alpha_1 \beta_2 X_2} \sim F(\alpha_1, \alpha_2)$.

Proof. As was proven in *PQIC Mathematical Notes, Analytical Series, Number 76: Univariate Continuous Gamma Distribution Random Variate Generation Using Integral Transform Methodology*, October 2020, we have

$$X_i \sim \Gamma\left(\frac{\alpha_i}{2}, \frac{\beta_i}{2}\right) \Longrightarrow \frac{\beta_i}{\alpha_i} X_i \sim \Gamma\left(\frac{\alpha_i}{2}, \frac{\alpha_i}{2}\right), \, i = 1, 2$$

so that if $Y_i = \frac{\beta_i}{\alpha_i} X_i$, $i = 1, 2$, then by the independence of the $X_i > 0$, the $Y_i > 0$ are also independent, which means

$$\begin{aligned}
P\left(\frac{\alpha_2 \beta_1 X_1}{\alpha_1 \beta_2 X_2} \leq t > 0\right) &= \int_0^t P\left(\frac{Y_1}{Y_2} = x > 0\right) dx \\
&= \int_0^t \int_0^\infty P\left(Y_1 = xy_2 \,|\, Y_2 = y_2\right) P\left(Y_2 = y_2\right) \frac{d\left(xy_2\right)}{dx} \, dy_2 \, dx \\
&= \int_0^t \int_0^\infty y_2 P\left(Y_1 = xy_2\right) P\left(Y_2 = y_2\right) \, dy_2 \, dx \\
&= \int_0^t \int_0^\infty y_2 \left(\frac{\left(\frac{\alpha_1}{2}\right)^{\frac{\alpha_1}{2}}}{\Gamma\left(\frac{\alpha_1}{2}\right)} \left(xy_2\right)^{\frac{\alpha_1}{2}-1} e^{-\frac{\alpha_1}{2}(xy_2)}\right) \left(\frac{\left(\frac{\alpha_2}{2}\right)^{\frac{\alpha_2}{2}}}{\Gamma\left(\frac{\alpha_2}{2}\right)} y_2^{\frac{\alpha_2}{2}-1} e^{-\frac{\alpha_2}{2} y_2}\right) dy_2 \, dx \\
&= \frac{\left(\frac{\alpha_1}{2}\right)^{\frac{\alpha_1}{2}} \left(\frac{\alpha_2}{2}\right)^{\frac{\alpha_2}{2}}}{\Gamma\left(\frac{\alpha_1}{2}\right) \Gamma\left(\frac{\alpha_2}{2}\right)} \int_0^t x^{\frac{\alpha_1}{2}-1} \int_0^\infty y_2^{\frac{\alpha_1+\alpha_2}{2}-1} e^{-\left(\frac{\alpha_1}{2}x + \frac{\alpha_2}{2}\right) y_2} \, dy_2 \, dx \\
&= \left(\begin{array}{c} \left(\frac{\left(\frac{\alpha_1}{2}\right)^{\frac{\alpha_1}{2}} \left(\frac{\alpha_2}{2}\right)^{\frac{\alpha_2}{2}}}{\Gamma\left(\frac{\alpha_1}{2}\right)\Gamma\left(\frac{\alpha_2}{2}\right)}\right) \int_0^t \frac{x^{\frac{\alpha_1}{2}-1}}{\left(\frac{\alpha_1}{2}x + \frac{\alpha_2}{2}\right)^{\frac{\alpha_1+\alpha_2}{2}}} \\ \times \int_0^\infty v^{\frac{\alpha_1+\alpha_2}{2}-1} e^{-v} \, dv \, dx \end{array}\right), \left(\begin{array}{c} v = \left(\frac{\alpha_1}{2}x + \frac{\alpha_2}{2}\right) y_2 \\ dy_2 = \frac{dv}{\frac{\alpha_1}{2}x + \frac{\alpha_2}{2}} \end{array}\right) \\
&= \frac{\left(\frac{\alpha_1}{2}\right)^{\frac{\alpha_1}{2}} \left(\frac{\alpha_2}{2}\right)^{\frac{\alpha_2}{2}}}{\Gamma\left(\frac{\alpha_1}{2}\right) \Gamma\left(\frac{\alpha_2}{2}\right)} \int_0^t x^{\frac{\alpha_1}{2}-1} \left(\frac{\alpha_1}{2}x + \frac{\alpha_2}{2}\right)^{-\frac{\alpha_1+\alpha_2}{2}} \Gamma\left(\frac{\alpha_1+\alpha_2}{2}\right) dx \\
&= \frac{\Gamma\left(\frac{\alpha_1+\alpha_2}{2}\right)}{\Gamma\left(\frac{\alpha_1}{2}\right) \Gamma\left(\frac{\alpha_2}{2}\right)} \left(\left(\frac{\alpha_1}{2}\right)^{\frac{\alpha_1}{2}} \left(\frac{\alpha_2}{2}\right)^{\frac{\alpha_2}{2} - \frac{\alpha_1+\alpha_2}{2}}\right) \int_0^t x^{\frac{\alpha_1}{2}-1} \left(1 + \frac{\alpha_1}{\alpha_2}x\right)^{-\frac{\alpha_1+\alpha_2}{2}} dx \\
&= \frac{\Gamma\left(\frac{\alpha_1+\alpha_2}{2}\right)}{\Gamma\left(\frac{\alpha_1}{2}\right) \Gamma\left(\frac{\alpha_2}{2}\right)} \left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{\alpha_1}{2}} \int_0^t x^{\frac{\alpha_1}{2}-1} \left(1 + \frac{\alpha_1}{\alpha_2}x\right)^{-\frac{\alpha_1+\alpha_2}{2}} dx
\end{aligned}$$

so that

$$P\left(\frac{\alpha_2 \beta_1 X_1}{\alpha_1 \beta_2 X_2} = t > 0\right) = \frac{\Gamma\left(\frac{\alpha_1+\alpha_2}{2}\right)}{\Gamma\left(\frac{\alpha_1}{2}\right) \Gamma\left(\frac{\alpha_2}{2}\right)} \left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{\alpha_1}{2}} t^{\frac{\alpha_1}{2}-1} \left(1 + \frac{\alpha_1}{\alpha_2}t\right)^{-\frac{\alpha_1+\alpha_2}{2}}$$

■

Remark 4 *Note that this result does not depend on the specific values of* β_i, $i = 1, 2$, *so that convenient values (such as* $\beta_1 = 2 = \beta_2$*) may be used at the discretion of the implementing analyst. In that particular convenient case, we would have*

$$X_i \sim \Gamma\left(\frac{\alpha_i}{2} > 0, 1\right) \Longrightarrow \frac{\alpha_2 X_1}{\alpha_1 X_2} \sim F(\alpha_1, \alpha_2) \;\; \textit{and} \;\; \frac{\alpha_1 X_2}{\alpha_2 X_1} \sim F(\alpha_2, \alpha_1)$$

Algorithm 5 *The following steps generates* N *random variates (with precision* tol*) from the univariate continuous* F *distribution with* r *and* s *degrees of freedom.*

1. Set $A = \varnothing$ and $C = 1$.
2. Generate N random variates $\boldsymbol{R}$ from the $\Gamma\left(\frac{r}{2}, 1\right)$ distribution with precision tol.[2]
3. Generate N random variates $\boldsymbol{S}$ from the $\Gamma\left(\frac{s}{2}, 1\right)$ distribution with precision tol.

[2] Note this is a special case of the general analytical methods used to generate random variates from a $\Gamma(\alpha, \beta)$ distribution as documented in *PQIC Mathematical Notes, Analytical Series, Number 76: Univariate Continuous Gamma Distribution*

4. Annex the value of $\frac{s\boldsymbol{R}[C]}{r\boldsymbol{S}[C]}$ to the Acceptance Set A.
5. Add 1 to C.
6. If $C = N + 1$ then skip to Step 7; otherwise skip to Step 4.
7. Return A as the set of N-many random variates with precision *tol* from the $F(r, s)$ distribution.

Note that this algorithm does not use any asymptotic estimation for "large" parameter values (see next section). It shall be PQICSTAT™ policy to leave the use of such alternative generation methods, e.g., decide when a parameter value is "large," to the discretion of the implementing analyst.

3. Simplifying Expression When r Or s Becomes Large

It may be a practical consideration to use asymptotic inference when r or s becomes large, as the calculation of the components of the density function may be impractical. For example, for an $F(4, 10000)$ distribution, even though $\frac{\Gamma(5002)}{\Gamma(2)\Gamma(5000)}\left(\frac{4}{10000}\right)^2$ admits to significant simplification, there is still a need to calculate $\left(1+\frac{4}{10000}x\right)^{-5002}$, which, even for large x, such as $x = 100$, only has significant digits starting in position 86. And this situation also applies when r is large relative to s, such as for an $F(10000, 4)$ distribution, for then $\left(1+\frac{10000}{4}x\right)^{-5002}$ must still be evaluated.

Therefore, the following claims provide an analytical means for evaluating such F distribution random variates by asymptotic methods. See also the appendix where the relationship between the Student-T and F distributions is explicitly demonstrated.

4. When s Becomes Large

Claim 6 *If $X \sim F(r, s)$, then $\lim_{s\to\infty} rX \sim \chi^2_r$.*

Proof. We have

$$\begin{aligned}\lim_{s\to\infty}\left(1+\frac{1}{s}u\right)^{-\frac{r+s}{2}} &= \left(\lim_{s\to\infty}\left(1+\frac{1}{s}u\right)^{-s}\right)^{\frac{1}{2}}\lim_{s\to\infty}\left(1+\frac{1}{s}u\right)^{-\frac{r}{2}}\\ &= \left(e^{-u}\right)^{\frac{1}{2}}(1)^{-\frac{r}{2}}\\ &= e^{-\frac{u}{2}}\end{aligned}$$

and using Stirling's Approximation, we have

$$\begin{aligned}\lim_{s\to\infty}\frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\left(\frac{1}{s}u\right)^{\frac{r}{2}} &= u^{\frac{r}{2}}\lim_{s\to\infty}\frac{\Gamma\left(\frac{r+s}{2}\right)}{s^{\frac{r}{2}}\Gamma\left(\frac{s}{2}\right)}\\ &= u^{\frac{r}{2}}\lim_{s\to\infty}\frac{\sqrt{2\pi\left(\frac{r+s}{2}-1\right)}\left(\frac{\frac{r+s}{2}-1}{e}\right)^{\frac{r+s}{2}-1}}{s^{\frac{r}{2}}\sqrt{2\pi\left(\frac{s}{2}-1\right)}\left(\frac{\frac{s}{2}-1}{e}\right)^{\frac{s}{2}-1}}\\ &= u^{\frac{r}{2}}\lim_{s\to\infty}e^{-\frac{r}{2}}\frac{\sqrt{\frac{r+s-2}{s-2}}\left(\frac{r+s}{2}-1\right)^{\frac{r+s}{2}-1}}{s^{\frac{r}{2}}\left(\frac{s}{2}-1\right)^{\frac{s}{2}-1}}\\ &= u^{\frac{r}{2}}e^{-\frac{r}{2}}\lim_{s\to\infty}\sqrt{1+\frac{r}{s-2}}\left(\frac{\frac{r+s}{2}-1}{s}\right)^{\frac{r}{2}}\left(\frac{\frac{r+s}{2}-1}{\frac{s}{2}-1}\right)^{\frac{s}{2}-1}\end{aligned}$$

Random Variate Generation Using Integral Transform Methodology, October 2020. Since $\beta = 1$ in this case, we have

$$f_{\Gamma\left(\frac{r}{2},1\right)}(x) = \frac{1}{\Gamma\left(\frac{r}{2}\right)}x^{\frac{r}{2}-1}e^{-x},\ x > 0$$

This simplifies to

$$\frac{x^{\frac{r}{2}-1}e^{-x}}{\left(\frac{r}{2}-1\right)!}\ (r\text{ even})\quad\text{and}\quad\frac{x^{\frac{r}{2}-1}e^{-x}}{\sqrt{\pi}\left(\frac{r}{2}-1\right)\left(\frac{r}{2}-2\right)\cdots\frac{1}{2}}(r\text{ odd})$$

Nevertheless, it shall be PQICSTAT™ policy to use the analytical methods in *op. cit.* to generate the gamma distribution random variates even in this special case.

$$= u^{\frac{r}{2}} e^{-\frac{r}{2}} \lim_{s\to\infty}\left(\sqrt{1+\frac{r}{s-2}}\right) \lim_{s\to\infty}\left(\frac{r+s-2}{2s}\right)^{\frac{r}{2}} \lim_{s\to\infty}\left(\frac{r+s-2}{s-2}\right)^{\frac{s}{2}-1}$$

$$= u^{\frac{r}{2}} \frac{1}{2^{\frac{r}{2}}} e^{-\frac{r}{2}} (1) \left(\lim_{s\to\infty}\left(1+\frac{r-2}{s}\right)^{\frac{r}{2}}\right) \lim_{s\to\infty}\left(1+\frac{r}{(2s+2)-2}\right)^{\frac{2s+2}{2}-1}$$

$$= \left(\frac{u}{2}\right)^{\frac{r}{2}} e^{-\frac{r}{2}} (1)(1) \lim_{s\to\infty}\left(1+\frac{\frac{r}{2}}{s}\right)^{s}$$

$$= \left(\frac{u}{2}\right)^{\frac{r}{2}} e^{-\frac{r}{2}} e^{\frac{r}{2}}$$

$$= \left(\frac{u}{2}\right)^{\frac{r}{2}}$$

$$\lim_{s\to\infty} P\left(rF(r,s) \le x\right) = \lim_{s\to\infty} P\left(F(r,s) \le \frac{x}{r}\right)$$

$$= \lim_{s\to\infty} \int_0^{\frac{x}{r}} \frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)} \left(\frac{r}{s}\right)^{\frac{r}{2}} t^{\frac{r}{2}-1} \left(1+\frac{r}{s}t\right)^{-\frac{r+s}{2}} dt$$

$$= \lim_{s\to\infty} \int_0^{x} \frac{1}{r} \frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)} \left(\frac{r}{s}\right)^{\frac{r}{2}} \left(\frac{u}{r}\right)^{\frac{r}{2}-1} \left(1+\frac{1}{s}u\right)^{-\frac{r+s}{2}} du, \quad \left(\begin{array}{c} u = rt, \\ dt = \frac{1}{r}\, du \end{array}\right)$$

$$= \int_0^{x} \frac{1}{u\Gamma\left(\frac{r}{2}\right)} \left(\lim_{s\to\infty} \frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \left(\frac{1}{s}u\right)^{\frac{r}{2}}\right) \left(\lim_{s\to\infty}\left(1+\frac{1}{s}u\right)^{-\frac{r+s}{2}}\right) du$$

$$= \int_0^{x} \frac{1}{u\Gamma\left(\frac{r}{2}\right)} \left(\left(\frac{u}{2}\right)^{\frac{r}{2}}\right) \left(e^{-\frac{u}{2}}\right) du$$

$$= \frac{1}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)} \int_0^{x} u^{\frac{r}{2}-1} e^{-\frac{u}{2}} du$$

which means

$$\lim_{s\to\infty} P\left(rF(r,s) = x\right) = \frac{1}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)} x^{\frac{r}{2}-1} e^{-\frac{x}{2}}$$

and this is the density function for a χ_r^2 distribution. ■

This proof also shows that the rate of convergence of this approximation depends on the two limits given by

$$\lim_{s\to\infty} \frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \left(\frac{1}{s}u\right)^{\frac{r}{2}} = \left(\frac{u}{2}\right)^{\frac{r}{2}} \quad \text{and} \quad \lim_{s\to\infty}\left(1+\frac{1}{s}u\right)^{-\frac{r+s}{2}} = e^{-\frac{u}{2}}$$

The first limit depends on r and u, while the second limit only depends on u – the particular value of r is not relevant to the limit, but it is to the rate of convergence. The quicker these terms converge, the closer the chi-square approximation comes to the actual F distribution value. However, the complicated interplay between these factors means such approximations must be considered “close” to the target distribution without claiming, or relying on, the false premise that the approximating value are actually coming from an F distribution.

5. When r Becomes Large

Corollary 7 *If* $X \sim F(r,s)$*, then* $\lim_{r\to\infty} s\frac{1}{X} \sim \chi_s^2$.

Proof. First note that

$$X \sim F(r,s) \Longrightarrow \frac{1}{X} \sim F(s,r)$$

[Proof: If $X \sim F(r,s)$, then

$$P(X \le x) = \frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)} \left(\frac{r}{s}\right)^{\frac{r}{2}} \int_0^{x} t^{\frac{r}{2}-1} \left(1+\frac{r}{s}t\right)^{-\frac{r+s}{2}} dt$$

so that

$$
\begin{aligned}
P\left(\frac{1}{X}\le x\right) &= P\left(X\ge\frac{1}{x}\right)\\
&= 1-P\left(X\le\frac{1}{x}\right)\\
&= 1-\frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)}\left(\frac{r}{s}\right)^{\frac{r}{2}}\int_0^{\frac{1}{x}} t^{\frac{r}{2}-1}\left(1+\frac{r}{s}t\right)^{-\frac{r+s}{2}}\,dt\\
&= 1-\frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)}\left(\frac{r}{s}\right)^{\frac{r}{2}}\int_x^{\infty} u^{-1-\frac{r}{2}}\left(1+\frac{r}{su}\right)^{-\frac{r+s}{2}}\,du,\quad\left(\begin{array}{c} u=\frac{1}{t}\\ dt=-\frac{du}{u^2}\end{array}\right)\\
&= 1-\frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)}\left(\frac{r}{s}\right)^{\frac{r}{2}}\int_x^{\infty} u^{-1-\frac{r}{2}}\left(\frac{r}{su}\right)^{-\frac{r+s}{2}}\left(\frac{s}{r}u+1\right)^{-\frac{r+s}{2}}\,du\\
&= 1-\frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)}\left(\frac{s}{r}\right)^{\frac{s}{2}}\int_x^{\infty} u^{\frac{s}{2}-1}\left(1+\frac{s}{r}u\right)^{-\frac{r+s}{2}}\,du\\
&= \frac{\Gamma\left(\frac{r+s}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)}\left(\frac{s}{r}\right)^{\frac{s}{2}}\int_0^{x} u^{\frac{s}{2}-1}\left(1+\frac{s}{r}u\right)^{-\frac{r+s}{2}}\,du
\end{aligned}
$$

so that $\frac{1}{X}\sim F(s,r)$.]

Then by Claim 6, we have $\lim_{r\to\infty} s\frac{1}{X}\sim\chi_s^2$. ■

These claims may be used interchangeably by the implementing analyst as analytical circumstances require.

6. Special Case When r And s Are Even Positive Integers

It is possible to reduce the size of the values involved in generating random variates from an F distribution when r and s are even integers. In this case, both $r=2k$ and $s=2l$ are positive even integers, and the density function becomes

$$
\begin{aligned}
f_X(x;2k,2l) &= \frac{\Gamma\left(\frac{2k+2l}{2}\right)}{\Gamma\left(\frac{2k}{2}\right)\Gamma\left(\frac{2l}{2}\right)}\left(\frac{2k}{2l}\right)^{\frac{2k}{2}} x^{\frac{2k}{2}-1}\left(1+\frac{2k}{2l}x\right)^{-\frac{2k+2l}{2}}\\
&= l\frac{(k+l-1)!}{(k-1)!l!}\left(\frac{k}{l}\right)^{k} x^{k-1}\left(1+\frac{k}{l}x\right)^{-(k+l)}\\
&= l\binom{k+l-1}{k-1}\left(\frac{k}{l}\right)^{k} x^{k-1}\left(1+\frac{k}{l}x\right)^{-(k+l)} \qquad (3)
\end{aligned}
$$

Furthermore, if $k=l$ in (3), then we have

$$f_X(x;2k,2k)=k\binom{2k-1}{k-1}x^{k-1}(1+x)^{-2k}$$

which means

$$\int_0^{\infty} x^{k-1}(1+x)^{-2k}\,dx=\left(k\binom{2k-1}{k-1}\right)^{-1}$$

and if $l=k-1$ in (3), then we have

$$f_X(x;2(l+1),2l)=l\binom{2l}{l}\left(1+l^{-1}\right)^{l+1}x^{l}\left(1+\left(1+l^{-1}\right)x\right)^{-(2l+1)}$$

We also have a special convenience if $r=2^w$ and $s=2^q$ for positive integers w and q, since then (3) may be applied iteratively until $f_X(x;2^w,2^q)$ is expressed only in terms of w and q.

7. MAPLE Implementation

The following MAPLE® module implements the analytical methods documented in this memorandum.

```
GenFRV:=proc(r,s,N,dg,M)
   local ii,F::Vector,R::Vector,S::Vector,val;
   description "Generates N Random Variates From F Distribution";
   options ‘Copyright 2020 PQI Consulting All Rights Reserved‘;
   Digits:=dg;
   R:=Vector(N); S:=Vector(N); F:=Vector(N);
   R:=GenGRV(r/2,1,N,dg,M); S:=GenGRV(s/2,1,N,dg,M);
   for ii from 1 to N do: F[ii]:=(s*R[ii])/(r*S[ii]); end do;
   return evalm(F);
end proc;
```

8. MMIX Implementation

The following MMIX code is the implementation of the MAPLE code of the previous section for Rational Arithmetic And Conversions (RAC) values. The reference to calculating univariate (continuous) standard normal random variates is found in *PQIC Mathematical Notes, Analytical Series, Number 77: Univariate Continuous Standard Normal Distribution Random Variate Generation Using Paired Methodology*, November 2020, and to calculating univariate (continuous) gamma distribution random variates is found in *PQIC Mathematical Notes, Analytical Series, Number 76: Univariate Continuous Gamma Distribution Random Variate Generation Using Integral Transform Methodology*, October 2020, *et seq.*. All other subroutine references are to the PQICSTAT™ Fundamental Instruction Set Operation Codes (FISOC) module library.

```
; PQICSTAT(tm) MMIX implementation for Generating Random Variates
;      From a Univariate Continuous F Distribution with Two
;      Parameter Degrees Of Freedom
;
; Application Specific Implementation Description

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; Version 1.0 Build 20240115A, et seq.

            PREFIX  :
t           IS      $255
Buf         IS      Data_Segment
NVAL        GREG    10                  Number of generated random variates
XVAL        GREG    520
```

MAPLE is a registered trademark of Maplesoft (a division of Waterloo Maple Inc.), 615 Kumpf Drive, Waterloo, Ontario, Canada, N2V 1K8. The MAPLE version used to produce the results found in this memorandum is 2017.1, June, 19, 2017, Maple Build ID 1238644.

```
RVAL        GREG    1040
cNVAL       BYTE    10
cXVAL       WYDE    520
cRVAL       WYDE    1040

            LOC     Buf
            GREG    @
BufZZ       OCTA    @+(256-@)&255

            LOC     BufZZ
cTWO        GREG    @
            BYTE    #1
            LOC     @+1
            WYDE    1
            LOC     @+4
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#2

            LOC     BufZZ+8*cXVAL
KONE        GREG    @
            BYTE    #1
            LOC     @+1
            WYDE    1
            LOC     @+4
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#1

            LOC     BufZZ+8*2*cXVAL
RDF         GREG    @           First F Degrees Of Freedom
            BYTE    #1
            LOC     @+1
            WYDE    1
            LOC     @+4
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0
            OCTA    #0,#0,#0,#0,#0,#0,#0,#0

```

```
                LOC     BufZZ+8*3*cXVAL
SDF             GREG    @           Second F Degrees Of Freedom
                BYTE    #1
                LOC     @+1
                WYDE    1
                LOC     @+4
                OCTA    #0,#0,#0,#0,#0,#0,#0,#0
                OCTA    #0,#0,#0,#0,#0,#0,#0,#0
                OCTA    #0,#0,#0,#0,#0,#0,#0,#0
                OCTA    #0,#0,#0,#0,#0,#0,#0,#0
                OCTA    #0,#0,#0,#0,#0,#0,#0,#0
                OCTA    #0,#0,#0,#0,#0,#0,#0,#0
                OCTA    #0,#0,#0,#0,#0,#0,#0,#0
                OCTA    #0,#0,#0,#0,#0,#0,#0,#0

                LOC     BufZZ+8*4*cXVAL
RMEM            GREG    @

                LOC     BufZZ+8*4*cXVAL+cNVAL*cRVAL
SMEM            GREG    @

                LOC     BufZZ+8*4*cXVAL+2*cNVAL*cRVAL
FMEM            GREG    @

                LOC     BufZZ+8*4*cXVAL+3*cNVAL*cRVAL
GFRVA           GREG    @

                LOC     BufZZ+8*4*cXVAL+(3*cNVAL+1)*cRVAL
GFRVB           GREG    @

                LOC     BufZZ+8*4*cXVAL+(3*cNVAL+2)*cRVAL
GFRVC           GREG    @

                LOC     BufZZ+8*4*cXVAL+(3*cNVAL+3)*cRVAL
GFRVD           GREG    @

                LOC     #2000
;               PREFIX  :pqicstat:
rRES            IS      $0

Main            SWYM    0
                SET     $1,:RDF     First F degree of freedom
                SET     $2,:SDF     Second F degree of freedom
                SET     $3,:FMEM    Destination of NVAL-many F(r,s) random variates
                PUSHJ   $0,:pqicstat:RAC:pqicGFRV:Start
Quit            TRAP    0,Halt,0

                PREFIX  :pqicstat:RAC:pqicGFRV:
rRA             IS      $0
rSB             IS      $1
rDEST           IS      $2
rNUM            IS      $3
rDEN            IS      $4
rSRR            IS      $5
```

```
rSSS        IS      $6
rMEMT       IS      $7
rMEMB       IS      $8
rCNT        IS      $9
rLMT        IS      $10
rCMP        IS      $11
rJ          IS      $12
rTMPA       IS      $13
rTMPB       IS      $14
rTMPC       IS      $15
rTMPD       IS      $16
rTMPE       IS      $17
rTMPF       IS      $18
rTMPG       IS      $19
rTMPAA      IS      $20
rTMPBB      IS      $21
rTMPCC      IS      $22
rTMPDD      IS      $23
rTMPEE      IS      $24
rTMPFF      IS      $25
rTMPGG      IS      $26

Start       GET     rJ,:rJ
            SET     rNUM,:RMEM
            SET     rDEN,:SMEM
            SET     rSRR,:GFRVA
            SET     rSSS,:GFRVB
            SET     rMEMT,:GFRVC
            SET     rMEMB,:GFRVD
            MULU    rLMT,:RVAL,:NVAL
            SET     rTMPB,rRA
            SET     rTMPC,:KONE
            SET     rTMPD,:cTWO
            SET     rTMPE,:KONE
            SET     rTMPF,rMEMT
            SET     rTMPG,rMEMB
            PUSHJ   rTMPA,:pqicstat:RAC:pqicRDIV:Start
            SET     rTMPB,rMEMT
            SET     rTMPC,rSRR
            PUSHJ   rTMPA,:pqicstat:RAC:pqicXCPX:Start
            SET     rTMPB,rMEMB
            ADDU    rTMPC,rSRR,:XVAL
            PUSHJ   rTMPA,:pqicstat:RAC:pqicXCPX:Start
            SET     rTMPB,rSB
            SET     rTMPC,:KONE
            SET     rTMPD,:cTWO
            SET     rTMPE,:KONE
            SET     rTMPF,rMEMT
            SET     rTMPG,rMEMB
            PUSHJ   rTMPA,:pqicstat:RAC:pqicRDIV:Start
            SET     rTMPB,rMEMT
            SET     rTMPC,rSSS
            PUSHJ   rTMPA,:pqicstat:RAC:pqicXCPX:Start
            SET     rTMPB,rMEMB
```

```
            ADDU    rTMPC,rSSS,:XVAL
            PUSHJ   rTMPA,:pqicstat:RAC:pqicXCPX:Start
            SET     rTMPB,rSRR
            ADDU    rTMPC,rSRR,:XVAL
            SET     rTMPD,:KONE
            SET     rTMPE,:KONE
            SET     rTMPF,rNUM          Numerator Gamma Random Variates
            ADDU    rTMPG,rNUM,:XVAL
            PUSHJ   rTMPA,:pqicstat:RAC:RACGenGRV:Start
            SET     rTMPB,rSSS
            ADDU    rTMPC,rSSS,:XVAL
            SET     rTMPD,:KONE
            SET     rTMPE,:KONE
            SET     rTMPF,rDEN          Denominator Gamma Random Variates
            ADDU    rTMPG,rDEN,:XVAL
            PUSHJ   rTMPA,:pqicstat:RAC:RACGenGRV:Start
            SET     rCNT,0
Cycle       ADDU    rTMPB,rNUM,rCNT
            ADDU    rTMPD,rDEN,rCNT
            ADDU    rTMPF,rDEST,rCNT
            ADDU    rCNT,rCNT,:XVAL
            ADDU    rTMPC,rNUM,rCNT
            SET     rTMPBB,rTMPB
            SET     rTMPCC,rTMPC
            SET     rTMPDD,rSB
            SET     rTMPEE,:KONE
            SET     rTMPFF,rMEMT
            SET     rTMPGG,rMEMB
            PUSHJ   rTMPAA,:pqicstat:RAC:pqicRMUL:Start
            SET     rTMPBB,rMEMT
            SET     rTMPCC,rSRR
            PUSHJ   rTMPAA,:pqicstat:RAC:pqicXCPX:Start
            SET     rTMPB,rSRR
            SET     rTMPBB,rMEMB
            SET     rTMPCC,rSSS
            PUSHJ   rTMPAA,:pqicstat:RAC:pqicXCPX:Start
            SET     rTMPC,rSSS
            ADDU    rTMPE,rDEN,rCNT
            SET     rTMPBB,rTMPD
            SET     rTMPCC,rTMPE
            SET     rTMPDD,rRA
            SET     rTMPEE,:KONE
            SET     rTMPFF,rMEMT
            SET     rTMPGG,rMEMB
            PUSHJ   rTMPAA,:pqicstat:RAC:pqicRMUL:Start
            SET     rTMPB,rSRR
            SET     rTMPC,rSSS
            SET     rTMPD,rMEMT
            SET     rTMPE,rMEMB
            SET     rTMPF,rDEST
            ADDU    rTMPG,rDEST,rCNT    F Random Variate Calculation
            PUSHJ   rTMPA,:pqicstat:RAC:pqicRDIV:Start
            ADDU    rCNT,rCNT,:XVAL
            CMPU    rCMP,rCNT,rLMT
```

```
             BN      rCMP,Cycle
   Quit      PUT     :rJ,rJ
             POP     0,0
```

9. Appendix: The Relationship Between The Student-T And F Distributions

When using the F distribution as the discriminator of choice in reducing linear models in designed experiments, it is commonly the case to use an $F(1,n)$ distribution, where the factor in question has only one degree of freedom with which to make a significance decision.

There is a close relationship between the (unscaled) Student-T distribution $t_n(0,1,1)$ density function and the $F(1,n)$ distribution density function. In particular, we have

$$P(t_n(0,1,1)=x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\left(\frac{1}{1+\frac{1}{n}x^2}\right)^{\frac{n+1}{2}}$$

and

$$\begin{aligned} P(F(1,n)=x) &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}\left(\frac{1}{n}\right)^{\frac{1}{2}} x^{\frac{1}{2}-1}\left(1+\frac{1}{n}x\right)^{-\frac{n+1}{2}} \\ &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\left(\frac{1}{\sqrt{x}}\right)\left(\frac{1}{1+\frac{1}{n}x}\right)^{\frac{n+1}{2}} \end{aligned} \tag{4}$$

so that

$$\begin{aligned} \frac{P(F(1,n)=x)}{P(t_n(0,1,1)=x)} &= \frac{\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)}\left(\frac{1}{\sqrt{x}}\right)\left(\frac{1}{1+\frac{1}{n}x}\right)^{\frac{n+1}{2}}}{\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)}\left(\frac{1}{1+\frac{1}{n}x^2}\right)^{\frac{n+1}{2}}} \\ &= \frac{1}{\sqrt{x}}\left(\frac{x^2+n}{x+n}\right)^{\frac{n+1}{2}} \end{aligned}$$

Since

$$\frac{x^2+n}{x+n} = 1 + x\frac{x-1}{x+n}$$

we have

$$\begin{aligned} \lim_{n\to\infty}\left(\frac{x^2+n}{x+n}\right)^{\frac{n+1}{2}} &= \sqrt{\lim_{n\to\infty}\frac{x^2+n}{x+n}}\sqrt{\lim_{n\to\infty}\left(\frac{x^2+n}{x+n}\right)^n} \\ &= \sqrt{\lim_{n\to\infty}\left(1+x\frac{x-1}{x+n}\right)^n} \\ &= \sqrt{\left(\frac{\lim_{n\to\infty}\left(1+x\frac{x-1}{x+n}\right)^{\frac{x+n}{x(x-1)}}}{\lim_{n\to\infty}\left(1+x\frac{x-1}{x+n}\right)^{\frac{x}{x(x-1)}}}\right)^{x(x-1)}} \\ &= \sqrt{e^{x(x-1)}} \\ &= e^{\frac{1}{2}x(x-1)} \end{aligned}$$

This means random variates from $t_n(0,1,1)$ may be generated at the same time as for $F(1,n)$ through

$$P(F(1,n)=x) = \frac{1}{\sqrt{x}}\left(\frac{x^2+n}{x+n}\right)^{\frac{n+1}{2}} P(t_n(0,1,1)=x)$$

Hence, if x_0 is a random variate from $t_n(0,1,1)$, then[3]

[3] It shall be PQICSTAT™ policy to calculate the value $x_0^* = f^{-1}_{F(1,n)}(y_0)$ through the analytical method `IBSFTI` documented in *PQIC Mathematical Notes, Analytical Series, Number 26: The Implicit Bracketing System For The Inverse*, October 2008.

$$x_0^* = f_{F(1,n)}^{-1}\left(\frac{1}{\sqrt{x_0}}\left(\frac{x_0^2+n}{x_0+n}\right)^{\frac{n+1}{2}} P\left(t_n\left(0,1,1\right)=x_0\right)\right)$$

is a random variate from $F\left(1,n\right)$, and if x_1 is a random variate from $F\left(1,n\right)$, then[4]

$$x_1^* \in f_{t_n(0,1,1)}^{-1}\left(\sqrt{x_1}\left(\frac{x_1+n}{x_1^2+n}\right)^{\frac{n+1}{2}} P\left(F\left(1,n\right)=x_1\right)\right) \tag{5}$$

is a random variate from $t_n\left(0,1,1\right)$. Note that a single x_1 most likely will produce a choice of two x_1^* values (one corresponding to the "upswing" part of the density function, and the other to the "downswing" part), one of which may be unbiasedly chosen at random; furthermore, since the density function of a $t_n\left(0,1,1\right)$ distribution is an even function, i.e., we have

$$P\left(t_n\left(0,1,1\right)=-x\right) = P\left(t_n\left(0,1,1\right)=x\right)$$

then if $x_1^{*(1)}$and $x_1^{*(2)}$ are the two values that come from (5), then

$$x_1^{*(1)} = -x_1^{*(2)}$$

However, since the density function of an $F\left(1,n\right)$ distribution is always decreasing, x_0^* is always unique. [Proof: From (4) we have

$$\begin{aligned}
\frac{d}{dx}P\left(F\left(1,n\right)=x>0\right) &\propto \begin{pmatrix} x^{-\frac{1}{2}}\frac{d}{dx}\left(1+\frac{1}{n}x\right)^{-\frac{n+1}{2}} \\ +\left(1+\frac{1}{n}x\right)^{-\frac{n+1}{2}}\frac{d}{dx}x^{-\frac{1}{2}} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{n+1}{2n}x^{-\frac{1}{2}}\left(1+\frac{1}{n}x\right)^{-\frac{n+3}{2}} \\ -\frac{1}{2}x^{-\frac{3}{2}}\left(1+\frac{1}{n}x\right)^{-\frac{n+1}{2}} \end{pmatrix} \\
&= -\frac{1}{2n}\left(\left(n+2\right)x+n\right)x^{-\frac{3}{2}}\left(1+\frac{1}{n}x\right)^{-\frac{n+3}{2}} \\
&< 0
\end{aligned}$$

since $n, x > 0$.]

We also have

$$\lim_{n\to\infty}\frac{P\left(F\left(1,n\right)=x\right)}{P\left(t_n\left(0,1,1\right)=x\right)} = \frac{e^{\frac{1}{2}x(x-1)}}{\sqrt{x}}$$

and since $\lim_{n\to\infty} P\left(F\left(1,n\right)=x\right) \sim \chi_1^2 = \frac{e^{-\frac{1}{2}x}}{\sqrt{2\pi x}}$ by Claim 6, we have

$$\lim_{n\to\infty} P\left(t_n\left(0,1,1\right)=x\right) = \sqrt{x}e^{-\frac{1}{2}x(x-1)}\lim_{n\to\infty} P\left(F\left(1,n\right)=x\right)$$

While the lower bound for this method is clearly 0, the upper bound must be found iteratively by increasing an argument x_c until $f_{F(1,n)}\left(x_c\right) < y_0$. This cycle ends when the first instance of such a value x_{c_*} is found. The bracketing interval to find x_0^* is then $[0, x_{c_*}]$. The incremental order of magnitude in the iterative step is at the discretion of the implementing analyst.

[4] It shall be PQICSTAT™ policy to find the positive value of $0 < x_1^* = f_{t_n(0,1,1)}^{-1}\left(y_0\right)$ through the analytical method `IBSFTI` documented in *op. cit.*, with a lower bound of 0 and upper bound found by the same iterative method as would be used for the $F\left(1,n\right)$ density function case. The lower bound is the peak value of this density function, which marks the value at which the inverse function is a single point. [Proof: We have

$$\begin{aligned}
0 = \frac{d}{dx}f_{t_n(0,1,1)}\left(x_p\right) &\propto \frac{d}{dx}\left(\frac{1}{1+\frac{1}{n}x_p^2}\right)^{\frac{n+1}{2}} \\
&= \frac{n+1}{2}\left(\frac{2x_p}{n}\right)\left(\frac{1}{1+\frac{1}{n}x_p^2}\right)^{\frac{n-1}{2}} \\
&= \left(1+\frac{1}{n}\right)x_p\left(\frac{1}{1+\frac{1}{n}x_p^2}\right)^{\frac{n-1}{2}} \\
&\Longrightarrow x_p = 0
\end{aligned}$$

since $1+\frac{1}{n}x^2 \geq 1 > 0$ for all $x \in \mathbb{R}$.

$$= \left(\sqrt{x}e^{-\frac{1}{2}x(x-1)}\right)\left(\frac{e^{-\frac{1}{2}x}}{\sqrt{2\pi x}}\right)$$
$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$$
$$\sim N(0,1)$$

so that asymptotically the $t_n(0,1,1)$ distribution agrees with the standard normal distribution (and may be used to approximate $t_n(0,1,1)$ for large n, although the convergence is especially slow for large x).

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